WHY NOT MAKE IT INTERESTING?

A vignette on the wording of problems.

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I recently found this problem on a math blog[1].

> **In the diagram below, quadrilaterals ABCD and ECFG are both squares. Points D, E and F are collinear. DE and EF are 4 cm and 6 cm, respectively. Determine the area of square ABCD.**
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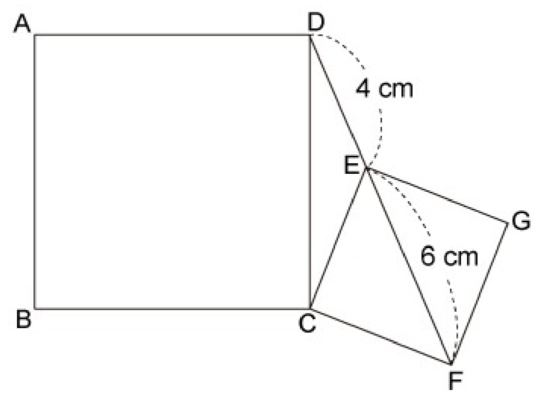
> 
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It's identified as one of the "five triangles problems" which I had never heard of, but the author commented that these problems often permit multiple solutions. The four solution methods that he included in the blog are 1) the law of cosines, 2) GeoGebra, 3) a trick, and then the Pythagorean Theorem, and 4) coordinate geometry[2].

So, here is my take on the problem.

This is a nice picture. It appeals to me but the picture is a bit cluttered. And why 4 and 6? Why not 2 and 3? Or better yet, a 1 somewhere? And where do the 4 and 6 come from? Given these numbers, I surmise that no other ratio (except 2:3) would work for these lengths. Why? The size of these squares is fairly arbitrary so what is the fundamental relationship that dictates this ratio?

And that, to me, is a much more interesting problem than calculating the area of a square.

In my work I think a lot about the way that problems are posed. Actually it's the focus of my thesis: the mathematical thinking that arises from the way problems are posed. The way that this problem is written is very redundant. Why include the picture if you are going to go into such detail in the description? In fact, given only the description, wouldn't students want to draw a picture?! What an opportunity – by focusing on the description they experience the power of definitions (squares, collinear), they have to negotiate how to place labels on a

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1 http://casmusings.wordpress.com/2013/10/06/two-squares-two-triangles-and-some-circles/

2 The blog post includes sketches of these solutions and comments posted by readers.

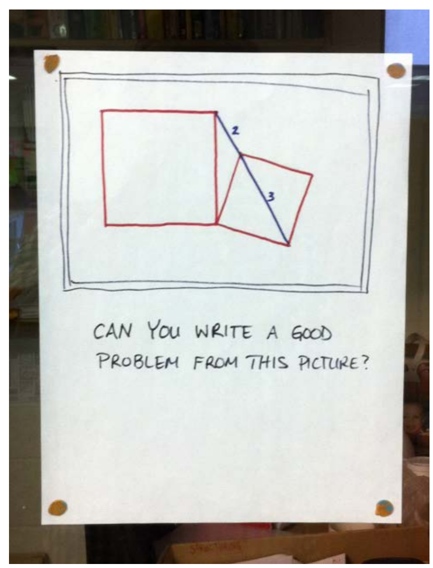
square (creating meaning in symbols) and because of this precision and care they can begin to feel ownership the problem, making it much more interesting to do. No, it's not easy, but wouldn't you prefer to be frustrated and interested rather than capable and bored?

On the other hand, why not just give the picture (pared-down with no labels[3] on the vertices) and pose the problem as, "What is interesting about this picture?" This, I believe, is an even better problem because it gives even more ownership to the students. In a classroom setting where they can discuss with their peers, the debate can become lively and quite enjoyable. Of course, for problems which are stated in this way, "Say what you see" (John Mason[4] uses these words) it is nice to follow up with a more directed activity; Fawcett[5] used the following phrasing:

> State all the properties of the figure that you are willing to accept. Then, give a complete argument justifying why you believe your assertions to be correct.

Sure, this is a friendlier version of "propose and prove", but it *is* friendlier. The presentation of this and many other textbook-type exercises seems to imply, "We know something that you don't know and we're not going to tell you. Figure it out." It's no wonder that most students don't want to do it. It's psychologically unappealing. But maybe it's just that textbook font that is making me feel rebellious. And GeoGebra? Really. Only if you take pleasure in inefficiency.

What I am saying is that instead of alternate solutions to the same problem, maybe it's pedagogically more interesting to consider alternate *questions* of the same problem. We already have a raft of things that are mathematically interesting to work with. Can't we pose these problems in a way that grabs students' interest in the *objects and relationships at the heart of the problem* rather than trying to tempt them with the procedures used to understand these things?



Can we inspire curiosity by putting this problem differently?

My office at the university is in a common area among many mathematics graduate students and so I posted this problem on my office window.

Well, two of my fellow graduate students soon approached me with sketches and ideas.

Graduate 1 (PhD student) made the observation that (In the terms provided with the first picture[6]) *A*, *E*, and *G* are also collinear. From that he wondered, "What is the length of that line?" Well, that really got him thinking about the size of the object: "What is the diameter of this whole shape?" And this was a good problem for him. He solved it by placing the shape in a complex coordinate system with *F* at the origin and calculating the distance from *F* to *A*.

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[3] Why force labels? It's asking students to discuss something using someone else's language.
[4] Mason, J. & Johnston-Wilder, S. (2006). *Designing and Using Mathematical Tasks,* 2nd ed. St. Albans: Tarquin.
[5] Fawcett, H.P. (1938). *The nature of proof* (1938 Yearbook of the National Council of Teachers of Mathematics).New York: Columbia University Teachers College Bureau of Publications.
[6] This is why we use notation –so that we can talk about things explicitly. But only then do we need it!

Graduate 2 (Master's student) came to me with the following problem, but in the course of our discussion he came up with another question which I think you will agree is superb. The first problem he brought to me had a few parts (I paraphrase):

a. Find the dimensions of the small square (the length a=CF).
b. Find the angle α=DCE. (He used the law of cosines and the law of sines.)
c. Hold the large square fixed and vary the angle α. As α changes, the size of the small square changes. For what values of α and a is the area of triangle A maximal? When this happens, what are the values of DE and EF (originally 4 and 6).

This is a good ol' standard-type math problem where you're led by the hand through some procedures and in the end is some amazing result that you are expected to be delighted with. Right. He admitted that as it was written, the problem wasn't really all that interesting. I mean: But there is an interesting idea contained here! So why not just ask it outright? "What happens when you rotate these squares out?" or "pivot at C" or whatever words you like. In fact, don't you think that if students were given just the picture, or maybe just the description, this question might just come up naturally? Maybe.

But then we talked a little more… and as he was talking about the smaller square getting bigger as the two squares rotated in together, he noticed that the point E would eventually meet point D (α=0 and we have two equal squares sitting right next to each other). Of course, going the other way, as α gets bigger and bigger, the point E eventually meets point C and the small square shrinks to a point. At this he sat back, "That's interesting because the size of the square just shrinks but one of its vertices traces some arc from D to C. I wonder if it's part of a circle. And then…

"Actually, point F is even more interesting. As α varies, the path it traces as looks like a spiral."

What profound and interesting results! What a world this opens up!

I have seen it happen, with only a few weeks of resistance, in my very own class. Undergraduate students are capable of mathematical creativity too. We need to work on posing those standard, well-known problems in ways that encourage them to be curious, to be interested.

And what a joy for students to be able to experience the creative part of mathematics: the *process of mathematical thinking* rather than the product of mathematical thought[7].

[7] Richard Skemp lamented that this is what happens when we teach mathematics in a logical procession. Skemp, R. R. (1971). *The psychology of learning mathematics*. Psychology Press. Baltimore, MD: Penguin Books.